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Stability of Pexiderized Functional Equation in Complex Banach Spaces: A Fixed-Point Approach

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Abstract

The study of Hyers-Ulam-Rassias stability for several functional equations has been wildly spreaded in the context of different areas of mathematics and such type stability in real Banach spaces along with its several extensions has been examined by a number of mathematicians. In this paper we prove the Hyers-Ulam-Rassias stability of a Pexiderized functional equation in complex Banach spaces under suitable conditions.

Keywords:

Hyers-Ulam stability;
Pexider type functional equation;
Complex Banach spaces;
Alternative fixed-point theorem.

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1. Introduction

The idea of stability of functional equations, in particular, about the stability of group homomorphism, was first posed by Ulam [15] and it was first partially answered by Hyers [7] for Banach spaces. Aoki [1] generalized the result of Hyers for additive mapping and it was further generalized by Rassias [14] for linear mappings by considering an unbounded Cauchy difference. A generalization of the Rassias theorem was obtained by Gavruta [6] by replacing the unbounded Cauchy difference with a general control function in the spirit of Rassias' approach. A number of outcomes regarding the stability problems of various functional equations have been extensively investigated by a number of researchers [3, 5, 8, 9, 10, 11].

In this paper we prove the Hyers-Ulam-Rassias stability for Pexiderized functional equation $f(x+iy) = g(x) - h(y)$ in complex Banach spaces by using the fixed-point method and examine a property of the solution of the above functional equation in technical terms.

2. Mathematical Background

First, we describe the notion of Pexiderized additive functional equation.

A mapping $f: R \to R$ is said to be an additive form if $f(x) = ax$ for all $x, a \in R$.

If X and Y are assumed to be a real vector space and a Banach space respectively then for a mapping $f: X \to Y$, consider a functional equation

$$f(x+y) = f(x) + f(y) \quad . \ . \ . \qquad (2.1).$$

which is known as the Cauchy functional equation and any solution of (2.1) is termed as an additive mapping. Particularly, if $X = Y = R$, the additive form $f(x) = ax$ is a solution of(2.1).

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The form

$$f(x+y)=g(x)+h(y)$$

is known as Pexiderized additive functional equation which is an extension of the above definition of additive functional equation. In this paper, we consider the Pexiderized additive functional equation in complex Banach spaces in the form

$$f(x+iy)=g(x)-h(y)\,.\;\;. \;\;. \qquad (2.2).$$

Now we describe a definition and a theorem which will be needed in the sequel.

Definition 2.1: Let X be a non-empty set. A function $d: X\times X\to[0,\infty]$ is called a generalized metric on X if d satisfies

(i) $d(x,y)=0$ if and only if $x=y$.

(ii) $d(x,y)=d(y,x)$ for all $x,y\in X$.

(iii) $d(x,y)\le d(x,z)+d(z,y)$ for all $x,y,z\in X$.

Then (X,d) is called a generalized metric space in short g. m. s.

Definition 2.2 [2]: Let (X,d) be a generalized metric space (g. m. s.), $\{x_n\}$ be a sequence in X, and $x\in, X$. We say that $\{x_n\}$ is g. m. s. convergent to x if and only if $d(x_n,\ x)\to, 0$ as $n\to\infty$. We denote this by $x_n\to x$.

Definition 2.3 [2]: Let(X,d), be a g. m. s. and $\{x_n\}$ be a sequence in X. We say that $\{x_n\}$ is Cauchy sequence if and only if for each $\varepsilon>0$,there exists a natural number N such that $d(x_n,x_m)<N$ for all $n>m>N$.

Definition 2.4 [2]: Let \(X,d) be a g. m. s. Then (X,d) is called a complete g. m. s. if every g. m. s. Cauchy seuqence is g. m. s. convergent in X.

Theorem: (The fixed point alternative theorem, [4])

Let (X,d) be a complete generalized metric space and let $J: X\to X$ be a strictly contractive mapping with Lipschitz constant $0<L<1$, that is,

$$d(Jx,Jy)\le Ld(x,y)$$

for all $x,y\in X$. Then for each $x\in X$, either

$$d(J^nx,J^nx)=\infty$$

or,

$$d(J^nx,J^{n+1}x)<\infty \quad \forall n\ge n_0$$

for some non-negative integers n_0. Moreover, if the second alternative holds then

(1) the sequence $\{J^nx\}$ converges to a fixed point $y^\star$ of J;

(2)$y^\star$ is the unique fixed point of J in the set

$$Y=\{y\in X: d(J^{n_0}x,y)<\infty\};$$

(3) $d(y,Y^\star)\le\left(\frac{1}{1-L}\right)d(y,Jy)$ for all $y\in Y$.

3. A Property of the function

Throughout this section, we consider that X and Y are complex vector spaces.

Result 3.1. Functions $f,g,h: X\to Y$ with $f(0)=0$ satisfies

$$f(x+iy)=g(x)-h(y)\,.\;\;.\;\;. \qquad (3.1)$$

for all $x,y\in X$, then the function $f: X\to Y$ is additive, that is,

$$f(x+y)=f(x)+f(y)$$

for all $x,y\in X$ and conversely if a mapping $f: X\to Y$ is additive, then the mapping $f: X\to Y$ satisfies (3.1).

Proof. Let $f: X\to Y$ satisfies (3.1).

Putting $x=y=0$ in (3.1) we get

$$f(0)=g(0)-h(0)$$

and putting $x=0$

$$f(iy)=g(0)-h(y)$$

also putting $y=0$

$$f(x)=g(x)-h(0).$$

Therefore

$$f(x+iy)=g(x)-h(y)$$

$$=f(x)+h(0)-g(0)+f(iy)=f(x)+f(iy)$$

since $f(0)=0$ implies $g(0)=h(0)$.

Now putting $iy=z$ then

$$f(x+z)=f(x)+f(z)$$

that is, f is additive.

Conversely, let a mapping $f: X\to Y$ additive, then

$$f(x+iy) = f(x) + f(iy)$$
$$= g(x) - h(0) + g(0) - h(y)$$
$$= g(x) - h(y)$$

for all $x, y \in X$. Hence $f : X \to Y$ satisfies (3.1).

4. The Hyers-Ulam-Rassias Stability

Theorem 4.1. Let X be a complex vector space and Y be a complex Banach space and $\phi : X \times X \to [\,0, \infty\,)$ be a mapping with $0 < L < 1$ such that $\phi\,(\,2x, -2ix\,) \le 2L\,\phi\,(\,x, -ix\,)$
and the mapping ϕ has the property $\phi\,(2^n x,\ 2^n y\,) < \infty$ also let $f : X \to Y$ be a mapping with $f(0) = 0$ and satisfying

$$\|f(x+iy) - g(x) + h(y)\| \le \phi(\,x, y\,)\ \ .\ .\ . \qquad (4.1)$$

for all $x, y \in X$.
Then there exists a unique additive mapping $A : X \to Y$ such that

$$\|f(x) - A(x)\| \le \frac{1}{2-2L}(\,\phi\,(\,x, -ix\,) + \phi(\,x, 0\,) + \phi\,(\,0, -ix\,))\ \ .\ .\ . \qquad (4.2)$$

Proof. For all $x, y \in X$, putting $x = 0$ in (4.1) we get

$$\|f(iy) - g(0) + h(y)\| \le \phi\,(\,0, y\,)\ \ .\ .\ . \qquad (4.3)$$

Also putting $y = 0$ in (4.1) we get

$$\|f(x) - g(x) + h(0)\| \le \phi\,(\,x, 0\,)\ \ .\ .\ . \qquad (4.4)$$

Therefore

$$\|f(x+iy) - f(x) - f(iy)\|$$
$$= \|\{\,f(x+iy) - g(x) + h(y)\} + \{\,g(x) - f(x) - h(0)\} - \{\,h(y) + f(iy) - g(0)\}\|$$
$$\le \phi\,(\,x, \quad y\,) + \phi\,(\,x, 0\,) + (\,0, y\,)\ \ .\ .\ . \qquad (4.5)$$

Hence putting $y = -ix$ in (4.5) we get

$$\|f(2x) - 2f(x)\| \le \phi\,(\,x, -ix) + \phi\,(\,x, 0\,) + \phi\,(\,0, -ix)\ \ .\ .\ . \qquad (4.6)$$

Consider a generalized metric on the set $S := \{\,g/g\!: X \to Y\,\}$ such that

$$d(\,g, h\,) = \inf\{\,c \in (\,o, \infty\,) :\ \|g(x) - h(x)\| \le c\,(\,\phi\,(\,x, -ix\,) + \phi(\,x, 0\,) + \phi(0, -ix\,))\,\}$$

for all $x \in X$.
It is easy to show that $(\,S, d\,)$ is a complete metric space.
Now we consider the mapping
$J : S \to S$ such that $J\,g(x) := \frac{1}{2}g(\,2x\,)$ for all $x \in X$.
For a fix $C \in (\,0, \infty\,)$ and $g, h \in S$ such that $d(\,g, h\,) < C$. Since

$$d(\,g, h\,) = \inf\{\,c \in (\,o, \infty\,) :\ \|g(x) - h(x)\| \le c\,(\,\phi\,(\,x, -ix\,) + \phi(\,x, 0\,) + \phi(0, -ix\,))\,\}$$

therefore, we find C_1 such that $d(\,g, h\,) \le C_1 < C$.
And $\|g(x) - h(x)\| \le C_1\big(\,\phi\,(\,x, -ix\,) + \phi(\,x, 0\,) + \phi(0, -ix\,)\big) < C\,\big(\,\phi\,(\,x, -ix\,) + \phi(\,x, 0\,) + \phi(0, -ix\,)\big)$
That implies $\left\|\frac{1}{2}g(2x) - \frac{1}{2}h(2x)\right\| \le \frac{C}{2}\big(\,\phi\,(\,2x, -i2x\,) + \phi(\,2x, 0\,) + \phi(0, -i2x\,)\big)$
That is,

$$\|Jg(x) - Jh(x)\| \le CL\ \ \big(\,\phi\,(\,x, -ix\,) + \phi(\,x, 0\,) + \phi(0, -ix\,)\big)$$

This implies that $d(\,Jg,\ Jh\,) \le L\,d\,(\,g, h\,)$ for all $g, h \in S$. That is, J is a strictly contractive mapping of S with the Lipschitz conatant L.
Also, from (4.6) we see that

$$d(\,f, Jf\,) \le \frac{1}{2}\ \ .\ .\ . \qquad (4.7)$$

Also $d(\,Jf, J^2\,f\,) \le L\,d(\,f, Jf\,) < \infty$ using (4.7)
Therefore, we find a natural number $n_0 = 1$ such that

$$d(J^n f, J^{n+1} f\,) = d\,(\,Jf, J^2 f\,) \le L\,d(f, Jf\,) < \infty$$

for all $n \ge n_0 = 1$.
So, by the fixed-point theorem on generalized metric space we obtain the existence of a mapping $A : X \to Y$ such that the following holds
1. A is a point of J, that is, $A(\,2x\,) = 2\,A(\,x\,)\ \ .\ .\ .$ (4.8)
for all $x \in X$.
Therefore, the mapping A is a unique fixed point of J in the set

$$E_1 = \{\,g \in E : d(J^{n_0} f, g\,) = d(\,Jf, g\,) < \infty\}$$

Thus $d(\,Jf, A\,) < \infty$.
Again from (4.7) $d(\,Jf, f\,) \le \frac{1}{2} < \infty$.

Thus $f \in E_1$.
Now, $d(f, A) \leq d(f, Jf) + d(Jf, A) < \infty$.
Thus, there exists $k \in (0, \infty)$ satisfying

$$\|f(x) - A(x)\| \leq k\left(\phi(x, -ix) + \phi(x, 0) + \phi(0, -ix)\right)$$

for all $x \in X$.
2. $d(J^n f, A) = \inf\{k \in (o, \infty) : \|J^n f(x) - A(x)\| \leq k(\phi(x, -ix) + \phi(x, 0) + \phi(0, -ix))\}$
Therefore $d(J^n f, A) \leq L^n C \to 0$ as $n \to \infty$ where $J^n f(x) = \frac{f(2^n x)}{2^n}$.
This implies $(\frac{f(2^n x)}{2^n}) \to A(x)$ as $n \to \infty$ . . . (4.9)
for all $x \in X$.
2. $d(f, A) \leq \frac{1}{1-L} d(f, Jf)$, which implies the inequality $d(f, A) \leq \frac{1}{2-2L}$.
Also replacing x and y by $2^n x$ and $2^n y$ respectively in (4.5) we obtain

$$\left\|\frac{f(2^n x + i2^n y)}{2^n} - \frac{f(2^n x)}{2^n} - \frac{f(i2^n y)}{2^n}\right\| \leq \frac{\phi(2^n x, 2^n y)}{2^n} + \frac{\phi(2^n x, 0)}{2^n} + \frac{\phi(0, 2^n y)}{2^n} \quad . . . \quad (4.10)$$

Taking limit as $n \to \infty$ the R. H. S. of (4.10) tends to zero as $n \to \infty$.
$A(x + iy) = A(x) + A(iy)$
Taking $iy = z$ we see that
$A(x + z) = A(x) + A(z)$
for all $x, z \in X$.
Thus, the mapping $A : X \to Y$ is an additive.
Corollary 4.2. Let $p < 1$ and θ be positive real number and $f : X \to Y$ be a mapping such that
$\|f(x + iy) - g(x) + h(y)\| \leq \theta(\|x\|^p + \|y\|^p)$
for all $x, y \in X$. Then there exists a unique additive mapping $A : X \to Y$ such that

$$\|f(x) - A(x)\| \leq \frac{4\theta\|x\|^p}{2 - 2^p}.$$

Proof. Define $\phi(x, y) = \theta(\|x\|^p + \|y\|^p)$ and $L = 2^{p-1}$ then the proof is followed by the Theorem 4.1.

Corollary 4.3. Let $p < 1$ and θ be positive real number and $f : X \to Y$ be a mapping such that

$$\|f(x + iy) - g(x) + h(y)\| \leq \theta(\|x\|^p . \|y\|^p)$$

for all $x, y \in X$. Then there exists a unique additive mapping $A : X \to Y$ such that

$$\|f(x) - A(x)\| \leq \frac{2\theta\|x\|^{2p}}{2 - 2^p}.$$

Proof. It can be proved in the similar way as Theorem 4.1 by defined $\phi(x, y) = \theta(\|x\|^p . \|y\|^p)$ and $L = 2^{p-1}$.

4. Conclusion

In this paper we have considered Pexiderized additive functional equation which is an extension of the additive functional equations in complex domain. As a future work we can take this functional equation in the field of fuzzy Banach spaces [13] and intuitionistic fuzzy Banach spaces [12] to examine the Hyers-Ulam-Rassias stability.

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